

# THE MAXIMUM SIZE OF A NON-TRIVIAL INTERSECTING UNIFORM FAMILY THAT IS NOT A SUBFAMILY OF THE HILTON–MILNER FAMILY

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**ABSTRACT.** The celebrated Erdős–Ko–Rado theorem determines the maximum size of a  $k$ -uniform intersecting family. The Hilton–Milner theorem determines the maximum size of a  $k$ -uniform intersecting family that is not a subfamily of the so-called Erdős–Ko–Rado family. In turn, it is natural to ask what the maximum size of an intersecting  $k$ -uniform family that is neither a subfamily of the Erdős–Ko–Rado family nor of the Hilton–Milner family is. For  $k \geq 4$ , this was solved (implicitly) in the same paper by Hilton–Milner in 1967. We give a different and simpler proof, based on the shifting method, which allows us to solve all cases  $k \geq 3$  and characterize all extremal families achieving the extremal value.

## 1. INTRODUCTION

Let  $X$  be an  $n$ -element set. For  $1 \leq k \leq n$ , let  $\binom{X}{k}$  denote the family of all subsets of  $X$  of cardinality  $k$ . A family  $\mathcal{F} \subseteq 2^X$  (or a hypergraph) is called *intersecting* if for all  $F_1, F_2 \in \mathcal{F}$ , we have  $F_1 \cap F_2 \neq \emptyset$ . A family  $\mathcal{F}$  is  *$k$ -uniform* if every member of  $\mathcal{F}$  contains exactly  $k$  elements. An intersecting family  $\mathcal{F}$  is *trivial* if  $\bigcap_{F \in \mathcal{F}} F \neq \emptyset$ , i.e., there is an element that is common to all members of  $\mathcal{F}$ . For families  $\mathcal{H} \subseteq 2^X$  and  $\mathcal{G} \subseteq 2^Y$ , we write  $\mathcal{H} \subseteq_S \mathcal{G}$  if there is an injective map  $f : X \rightarrow Y$  such that  $f(E) \in \mathcal{G}$  for every  $E \in \mathcal{H}$ . If both  $\mathcal{H} \subseteq_S \mathcal{G}$  and  $\mathcal{G} \subseteq_S \mathcal{H}$  hold, then we say that  $\mathcal{H}$  and  $\mathcal{G}$  are isomorphic. For simplicity, we often abuse notation and write  $\mathcal{H} = \mathcal{G}$  if  $\mathcal{H}$  and  $\mathcal{G}$  are isomorphic.

The celebrated Erdős–Ko–Rado theorem [5] determines the maximum size of a uniform intersecting family. For any  $x \in X$ , let  $\mathcal{F}(x)$  be the intersecting  $k$ -uniform family  $\{F \in \binom{X}{k} : x \in F\}$ . We call  $x$  the *center* of  $\mathcal{F}(x)$ . We write  $\mathcal{F}_0$  for any family isomorphic to  $\mathcal{F}(x)$ .

**Theorem 1.1** (The Erdős–Ko–Rado theorem [5]). *Let  $\mathcal{F}$  be a  $k$ -uniform intersecting family on  $X$  and suppose  $n \geq 2k$ . Then  $|\mathcal{F}| \leq \binom{n-1}{k-1}$ . If  $n > 2k$ , equality holds only for  $\mathcal{F}_0$ .*

For non-trivial intersecting families, Hilton and Milner [12] proved Theorem 1.2 below, whose statement is simplified if we introduce some notation first. For any  $k$ -set  $F \subset X$  and any  $x \in X \setminus F$ , let  $\mathcal{F}(F, x) = \{F\} \cup \{G \in \binom{X}{k} : x \in G, F \cap G \neq \emptyset\}$ . We call  $x$  the *center* of  $\mathcal{F}(F, x)$ . We write  $\mathcal{F}_1$  for any family isomorphic to such a family  $\mathcal{F}(F, x)$ . We also define the following families: for any 3-set  $S \subset X$ , let  $\mathcal{T}(S) := \{F \in \binom{X}{k} : |F \cap S| \geq 2\}$ . We write  $\mathcal{G}_2$  for any family isomorphic to

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such a family  $\mathcal{T}(S)$ . The celebrated *Hilton–Milner theorem* is as follows (for alternative proofs and generalizations, see, e.g., Borg [3], Frankl and Füredi [9] and Frankl and Tokushige [10]).

**Theorem 1.2** (The Hilton–Milner theorem [12]). *Let  $\mathcal{F}$  be a non-trivial  $k$ -uniform intersecting family on  $X$  with  $k \geq 2$  and  $n > 2k$ . Then  $|\mathcal{F}| \leq \binom{n-1}{k-1} - \binom{n-k-1}{k-1} + 1$ . Equality holds only for the family  $\mathcal{F}_1$  and, if  $k \in \{2, 3\}$ , for the family  $\mathcal{G}_2$ .*

The Hilton–Milner theorem determines the maximum size of an intersecting family *that is not EKR*, i.e., that is not contained in an  $\mathcal{F}_0$ . Note that the bound in the Hilton–Milner theorem is much smaller than the bound in the Erdős–Ko–Rado theorem (as long as  $k$  is not too large) and, therefore, the Hilton–Milner theorem shows the so-called “stability” of the Erdős–Ko–Rado theorem in a very strong sense (for other stability-type results related to the Erdős–Ko–Rado theorem, we refer the reader to Dinur and Friedgut [4], Keevash [15] and Keevash and Mubayi [16]). What lies beyond the 1967 theorem of Hilton and Milner, that is, beyond Theorem 1.2? Let us say that a family  $\mathcal{F}$  is *HM* if it is contained in the family  $\mathcal{F}_1$  or  $k \in \{2, 3\}$  and it is contained in  $\mathcal{G}_2$ . Our question is then the following:

**Question 1.3.** *What is the maximum size of an intersecting family  $\mathcal{H}$  that is neither EKR nor HM? Which families achieve the extremal value?*<sup>1</sup>

In fact, this question was partially answered in the 1967 paper of Hilton and Milner [12]. Their main result in that paper [12, Theorem 3], which contains Theorem 1.2 above, is as follows. Here, for simplicity, we state it only for  $k$ -uniform families.

**Theorem 1.4.** [12] *Fix integers  $\min\{3, s\} \leq k \leq n/2$  and let  $\mathcal{F} = \{A_1, \dots, A_m\}$  be a  $k$ -uniform intersecting family on  $X$ . Moreover, assume that for any  $S \subseteq [m]$  with  $|S| > m - s$ , we have*

$$\bigcap_{i \in S} A_i = \emptyset.$$

*Then*

$$m \leq \begin{cases} \binom{n-1}{k-1} - \binom{n-k}{k-1} + n - k & \text{if } 2 < k \leq s + 2, \\ \binom{n-1}{k-1} - \binom{n-k}{k-1} + \binom{n-k-s}{k-s-1} + s & \text{if } k \leq 2 \text{ or } k \geq s + 2. \end{cases} \quad (1.1)$$

*Moreover, the bounds in (1.1) are best possible.*

Theorem 1.4 contains the Erdős–Ko–Rado theorem as its special case  $s = 0$ , and it contains the Hilton–Milner theorem as its special case  $s = 1$ . Let us now consider Question 1.3. Suppose that  $k \geq 4$  and  $\mathcal{H}$  is neither EKR nor HM. Then  $\mathcal{H}$  satisfies the hypothesis of Theorem 1.4 for  $s = 2$ , and hence we know that  $|\mathcal{H}|$  is at most as large as specified in the second bound in (1.1). Unfortunately, when  $k = 3$ , Theorem 1.4 does not give a sharp bound. Moreover, Theorem 1.4 does not give any information about the extremal families that achieve the extremal values.

In this note we settle Question 1.3 completely. Let us start by noting that the case  $k = 2$  is trivial. Suppose that  $k = 2$  and that  $\mathcal{H}$  is not trivially intersecting. Then  $\mathcal{H}$  is a triangle, which means that  $\mathcal{H} = \mathcal{F}_1$  (in fact  $\mathcal{H} = \mathcal{F}_1 = \mathcal{G}_2$  when  $k = 2$ ). This means that, for  $k = 2$ , every

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<sup>1</sup>This question has also been asked on MathOverflow <http://mathoverflow.net/q/94438>.

intersecting family  $\mathcal{H}$  is either EKR or HM. We may therefore suppose that  $k \geq 3$  in what follows. Let us now describe the extremal families for our theorem.

**Definition 1.5** ( $\mathcal{G}(E, x_0)$ ,  $\mathcal{J}(E, J, x_0)$ ,  $\mathcal{G}_i$ ,  $\mathcal{J}_i$ ). For any  $i$ -set  $E \subseteq X$ , where  $2 \leq i \leq k$ , and any  $x_0 \in X \setminus E$ , we define the  $k$ -uniform family  $\mathcal{G}(E, x_0)$  on  $X$  by setting

$$\mathcal{G}(E, x_0) = \{G \in \binom{X}{k} : E \subseteq G\} \cup \{G \in \binom{X}{k} : x_0 \in G, G \cap E \neq \emptyset\}.$$

We write  $\mathcal{G}_i$  for any family isomorphic to a  $\mathcal{G}(E, x_0)$  as above. Now suppose  $1 \leq i \leq k-1$ . For any  $(k-1)$ -set  $E \subseteq X$ , any  $(i+1)$ -set  $J \subseteq X \setminus E$  and any  $x_0 \in J$ , we define the  $k$ -uniform family  $\mathcal{J}(E, J, x_0)$  on  $X$  by setting

$$\begin{aligned} \mathcal{J}(E, J, x_0) = \{G \in \binom{X}{k} : E \subseteq G, G \cap J \neq \emptyset\} \cup \{G \in \binom{X}{k} : J \subseteq G\} \\ \cup \{G \in \binom{X}{k} : x_0 \in G, G \cap E \neq \emptyset\}. \end{aligned}$$

We write  $\mathcal{J}_i$  for any family isomorphic to a  $\mathcal{J}(E, J, x_0)$  as above.

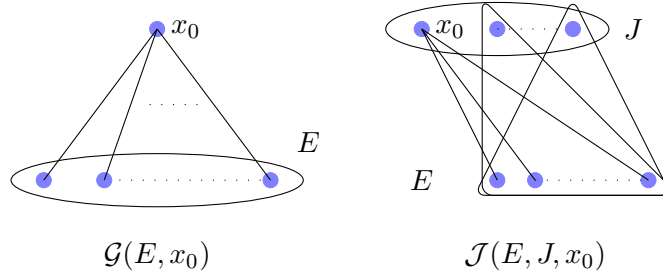


FIGURE 1. Extremal families:  $\mathcal{G}(E, x_0)$  and  $\mathcal{J}(E, J, x_0)$ . Each family consists of all  $k$ -sets that contain some pair (line segment) or set (ellipse or triangle) in the picture.

Note that  $\mathcal{J}_1 = \mathcal{F}_1 = \mathcal{G}_k$ , the Hilton–Milner family. We also remark that the families  $\mathcal{G}_i$  above appear as the extremal families in a result of Frankl [7] that generalizes the Hilton–Milner theorem. Our main result is as follows.

**Theorem 1.6.** Suppose  $k \geq 3$  and  $n > 2k$  and let  $\mathcal{H}$  be an intersecting  $k$ -uniform family on  $X$ . Furthermore, assume that  $\mathcal{H} \not\subseteq_S \mathcal{F}_0$ ,  $\mathcal{H} \not\subseteq_S \mathcal{F}_1$  and, if  $k = 3$ ,  $\mathcal{H} \not\subseteq_S \mathcal{G}_2$ . Then

$$|\mathcal{H}| \leq \binom{n-1}{k-1} - \binom{n-k-1}{k-1} - \binom{n-k-2}{k-2} + 2. \quad (1.2)$$

For  $k = 4$ , equality holds if and only if  $\mathcal{H} = \mathcal{J}_2$ ,  $\mathcal{G}_2$  or  $\mathcal{G}_3$ ; for every other  $k$ , equality holds if and only if  $\mathcal{H} = \mathcal{J}_2$ .

We now make some remarks on Theorem 1.6. Let us first of all mention that a quick calculation shows that the right-hand side of (1.2) is equal to the second bound in (1.1) with  $s = 2$ . We now note that Theorem 1.6 implies an Erdős–Ko–Rado type theorem with a maximum degree condition. For any family  $\mathcal{H}$  on  $X$  and  $x \in X$ , let  $d_{\mathcal{H}}(x)$  be the number of sets in  $\mathcal{H}$  containing  $x$ . Let  $d(\mathcal{H}) = \max_{x \in X} d_{\mathcal{H}}(x)$ . In [7], Frankl showed that, for any  $k$ -uniform intersecting family  $\mathcal{H}$  on  $[n]$

and  $2 \leq i \leq k$ , if  $d(\mathcal{H}) \leq d(\mathcal{G}_i)$ , then  $|\mathcal{H}| \leq |\mathcal{G}_i|$ . When  $k \geq 4$ , this generalizes the Hilton–Milner theorem because the case  $i = k$  is equivalent to the Hilton–Milner theorem. Theorem 1.6 also gives such a maximum degree form for  $k \geq 4$ .

**Corollary 1.7.** *Suppose  $k \geq 4$  and let  $\mathcal{H}$  be a  $k$ -uniform intersecting family on  $X$ . If  $d(\mathcal{H}) \leq d(\mathcal{J}_2)$ , then  $|\mathcal{H}| \leq |\mathcal{J}_2|$ . Moreover, if  $k \geq 5$ , equality holds if and only if  $\mathcal{H} = \mathcal{J}_2$ .*

*Proof.* If  $\mathcal{H}$  satisfies the assumptions of Theorem 1.6, then  $|\mathcal{H}| \leq |\mathcal{J}_2|$ . If  $\mathcal{H} \subseteq_S \mathcal{F}_0$  or  $\mathcal{H} \subseteq_S \mathcal{F}_1$ , then we have  $|\mathcal{H}| \leq d(\mathcal{H}) + 1 \leq d(\mathcal{J}_2) + 1 < |\mathcal{J}_2|$ . Clearly, if  $k \geq 5$  equality holds only if  $\mathcal{H}$  satisfies the assumptions of Theorem 1.6 and, therefore,  $\mathcal{H} = \mathcal{J}_2$ .  $\square$

For another Erdős–Ko–Rado type theorem with conditions on the maximum degree, see Füredi [11]. We also remark that there is another way of “going beyond” the Hilton–Milner theorem, namely, Frankl [6] investigated the maximum size of a  $k$ -uniform intersecting family whose transversal number (the minimum size of a vertex cover) is more than 2. We close by observing that “our way” of going beyond Hilton–Milner has very recently been considered by Jackowska, Polcyn and Ruciński [13], in a more general form (those authors define a certain hierarchy of Turán numbers; see [13]).

## 2. PROOF OF THE QUANTITATIVE PART OF THEOREM 1.6

We use the *shifting technique* in [5, 9]. Readers not familiar with shifting are strongly encouraged to study [8]. Here we give the definition of the shifting operator and briefly state some basic facts. To define shifting, we need to suppose that the elements of  $X$  are given some linear order. For  $x, y \in X$ ,  $x < y$ , we define  $S_{xy}(\mathcal{H}) = \{S_{xy}(E) : E \in \mathcal{H}\}$ , where

$$S_{xy}(E) = \begin{cases} (E \setminus \{y\}) \cup \{x\} & \text{if } x \notin E, y \in E, (E \setminus \{y\}) \cup \{x\} \notin \mathcal{H}, \\ E & \text{otherwise.} \end{cases}$$

**Proposition 2.1.** [5] *We have  $|S_{xy}(\mathcal{H})| = |\mathcal{H}|$ . Moreover,  $S_{xy}(\mathcal{H})$  is intersecting if  $\mathcal{H}$  is intersecting.*

Let  $\mathcal{H}$  be an intersecting  $k$ -uniform family on  $X = [n]$ . To prove Theorem 1.1, we apply the shifting operator  $S_{xy}$  repeatedly to  $\mathcal{H}$  for all  $1 \leq x < y \leq n$  until we get a *stable* family  $\mathcal{G}$ , i.e., such that  $S_{xy}(\mathcal{G}) = \mathcal{G}$  holds for all  $1 \leq x < y \leq n$ . We note that the shifting process must terminate and thus we always reach a stable family. Indeed, note that for any shift  $S_{xy}$  on  $\mathcal{H}$ , if  $S_{xy}(E) \neq E$ , then the sum of the elements in  $S_{xy}(E)$  is strictly smaller than the sum of the elements in  $E$  (because we replaced  $y$  by  $x$  and  $x < y$ ). Thus the sum, over all edges of  $\mathcal{H}$ , of all such sums strictly decreases unless  $S_{xy}(\mathcal{H}) = \mathcal{H}$ . So the shifting process must terminate.

We say a family  $\mathcal{H}$  is *EKR* (or *HM*) at  $x \in X$  if  $\mathcal{H} \subseteq_S \mathcal{F}_0$  (or  $\mathcal{H} \subseteq_S \mathcal{F}_1$ ) where  $x$  is mapped to the center of  $\mathcal{F}_0$  (or  $\mathcal{F}_1$ ). We say a family  $\mathcal{H}$  is *HM* at  $\{x, y, z\} \subseteq X$  if  $\mathcal{H} \subseteq_S \mathcal{G}_2$  where  $\{x, y, z\}$  is mapped onto the set  $\{x\} \cup E$  of  $\mathcal{G}_2$  in Definition 1.5.

Throughout this section, we will use the following fact, whose proof is trivial.

**Fact 2.2.** *Let  $\mathcal{H}$  be a  $k$ -uniform intersecting family on  $X$  and let  $x \in X$ . Then  $\mathcal{H}$  is neither EKR nor HM at  $x$  if and only if there are  $E, E' \in \mathcal{H}$  such that  $x \notin E$ ,  $x \notin E'$ .*

Let  $\mathcal{H}$  be an intersecting family of maximal size such that  $\mathcal{H} \not\subseteq_S \mathcal{F}_0$ ,  $\mathcal{H} \not\subseteq_S \mathcal{F}_1$  and if  $k = 3$ ,  $\mathcal{H} \not\subseteq_S \mathcal{G}_2$ . We prove the statement by induction on  $k \geq 2$ . The base case  $k = 2$  is trivial, because there is no such  $\mathcal{H}$ . For the case  $k = 3$  and  $n = 2k + 1 = 7$ , note that the HM family has size 13, which implies  $|\mathcal{H}| \leq 12$ , as desired (we will show the uniqueness of this case in Section 3.3). Hence, throughout this section, when  $k = 3$ , we may assume that  $n \geq 2k + 2 = 8$ .

We plan to apply repeatedly the shifting operator  $S_{xy}$  to  $\mathcal{H}$  for all  $1 \leq x < y \leq n$ . But we may fall into trouble if the family after shifting becomes a subfamily of  $\mathcal{F}_0$  or  $\mathcal{F}_1$  (or  $\mathcal{G}_2$  if  $k = 3$ ). We observe the following facts.

**Fact 2.3.** *For any  $\mathcal{H}' \not\subseteq_S \mathcal{F}_0$ ,  $\mathcal{H}' \not\subseteq_S \mathcal{F}_1$  and if  $k = 3$ ,  $\mathcal{H}' \not\subseteq_S \mathcal{G}_2$ , we have the following.*

- (i) *If  $S_{xy}(\mathcal{H}')$  is EKR (or HM) at some element, then  $S_{xy}(\mathcal{H}')$  is EKR (or HM) at  $x$ .*
- (ii) *If  $S_{xy}(\mathcal{H}')$  is HM at a 3-set  $A$ , then  $A = \{x, x_1, x_2\}$  for some  $x_1, x_2 \in X \setminus \{x, y\}$ .*

*Proof.* For any  $z \in X \setminus \{x\}$ , by Fact 2.2, there are at least two edges in  $\mathcal{H}'$  that do not contain  $z$ . Clearly, after the shift  $S_{xy}$ , these edges do not contain  $z$  as well. So  $S_{xy}(\mathcal{H}')$  is neither EKR nor HM at  $z$  and (i) follows.

For (ii), since  $\mathcal{H}'$  is not HM at any 3-set  $\{x_0, x_1, x_2\}$ , there exists an edge  $E \in \mathcal{H}'$  such that  $|E \cap \{x_0, x_1, x_2\}| \leq 1$ . If  $S_{xy}(\mathcal{H}')$  is HM at  $\{x_0, x_1, x_2\}$ , then  $|S_{xy}(E) \cap \{x_0, x_1, x_2\}| \geq 2$ . This happens only if one of  $\{x_0, x_1, x_2\}$  is  $x$  and none of them is  $y$ . We may assume that  $x_0 = x$  and thus (ii) follows.  $\square$

Note that  $\mathcal{G}_2$  is the extremal family in Theorem 1.2 only for  $k = 3$ . So when  $k \geq 4$ , if we get a family  $\mathcal{H}'$  such that  $S_{xy}(\mathcal{H}')$  is HM at a 3-set, then we continue the shifting process. By Fact 2.3, if we apply  $S_{xy}$  ( $x < y$ ) repeatedly to  $\mathcal{H}$ , then we obtain a family in one of the following four cases,

- (0) a family  $\mathcal{G}$  which is stable, i.e.,  $S_{xy}(\mathcal{G}) = \mathcal{G}$  holds for all  $x < y$ ,
- (1) a family  $\mathcal{H}_1$  such that  $S_{xy}(\mathcal{H}_1)$  is EKR at  $x$ ,
- (2) a family  $\mathcal{H}_2$  such that  $S_{xy}(\mathcal{H}_2)$  is HM at  $x$ , or
- (3) (for  $k = 3$  only) a family  $\mathcal{H}_3$  such that  $S_{xy}(\mathcal{H}_3)$  is HM at  $\{x, x_1, x_2\}$  for some  $x_1, x_2 \in X \setminus \{x, y\}$ .

In Cases (1) – (3), we will *not* apply the shift  $S_{xy}$  – otherwise we will get a family whose size is out of our control. Instead, we will adjust our shifting as shown in the following proposition.

**Proposition 2.4.** *In Case (i), for  $i = 1, 2, 3$ , there is a set  $X_i \subset X$  of size at most 4 such that for all  $x', y' \in X \setminus X_i$ ,  $x' < y'$ , we can apply (repeatedly) all the shifts  $S_{x'y'}$ , i.e., we will not be in any of Cases (1) – (3) and the resulting family  $\mathcal{G}$  satisfies that  $S_{x'y'}(\mathcal{G}) = \mathcal{G}$  for all  $x', y' \in X \setminus X_i$ ,  $x' < y'$ . Moreover,  $E \cap X_i \neq \emptyset$  for all  $E \in \mathcal{G}$  and when  $k \geq 4$ , the sets  $X_i$  can be chosen of size at most 3.*

*Proof.* We first assume  $k \geq 4$ . In Case (1), define  $X_1 = \{x, y\}$  and note that for any  $E \in \mathcal{H}_1$ ,  $E \cap X_1 \neq \emptyset$ . Indeed, for any  $E \in \mathcal{H}_1$ , since  $S_{xy}(E)$  contains  $x$ , we know that if  $x \notin E$ , then  $y \in E$ . In Case (2), observe that  $S_{xy}(\mathcal{H}_2)$  contains exactly one edge  $E_0 = \{z_1, \dots, z_k\}$  that does not contain  $x$ . Without loss of generality, assume that  $z_1 \neq y$  and let  $X_2 = \{x, y, z_1\}$ . Also note that for any  $E \in \mathcal{H}_2 \setminus \{E_0\}$ , we have  $E \cap \{x, y\} \neq \emptyset$ . Thus, for any  $E \in \mathcal{H}_2$ ,  $E \cap X_2 \neq \emptyset$ .

For  $i = 1, 2$ , apply repeatedly  $S_{x'y'}$  to the family for  $x' < y'$ ,  $x', y' \in X \setminus X_i$  and we claim that we will reach a family  $\mathcal{G}$  such that  $S_{x'y'}(\mathcal{G}) = \mathcal{G}$  for any  $x', y' \in X \setminus X_i$ ,  $x' < y'$ . Indeed, by Fact 2.2, it suffices to show that in each step, the current family  $\mathcal{H}'$  contains at least two  $k$ -sets that do not contain  $x'$ . Since for any  $E \in \mathcal{H}_i$ ,  $E \cap X_i \neq \emptyset$ , the maximality of  $|\mathcal{H}|$  implies that all  $k$ -sets containing  $X_i$  are in  $\mathcal{H}$  for  $i = 1, 2$ . Moreover, these sets stay fixed during the shifting process. This implies that there are at least  $\binom{n-3}{k-2}$  (if  $i = 1$ ) or  $\binom{n-4}{k-3}$  (if  $i = 2$ ) members of  $\mathcal{H}'$  that do not contain  $x'$  and we are done.

Now we prove the case  $k = 3$ . The following observation will be helpful.

**Fact 2.5.** *Given a 3-uniform family  $\mathcal{F}$  on  $X$  which is HM at a 3-set  $A$ , if there are at least three triples of  $\mathcal{F}$  containing both  $v, v' \in X$ , then  $\{v, v'\} \subseteq A$ .*

First assume that we reach Case (1). Let  $X_1 = \{x, y\}$  and note that for any  $E \in \mathcal{H}_1$ ,  $E \cap X_1 \neq \emptyset$ . We apply  $S_{x'y'}$  repeatedly to the family for  $x' < y'$ ,  $x', y' \in X \setminus X_1$ . By Fact 2.2, to show that we will *not* reach Cases (1) or (2) for any  $x' < y'$ ,  $x', y' \in X \setminus X_1$ , it suffices to show that in each step, the current family  $\mathcal{H}'$  contains at least two triples that do not contain  $x'$ . Since for any  $E \in \mathcal{H}_1$ ,  $E \cap X_1 \neq \emptyset$ , the maximality of  $|\mathcal{H}|$  implies that all the  $n - 2$  triples that contain  $X_1$  are in  $\mathcal{H}$  (so in  $\mathcal{H}'$ ). Moreover, these triples stay fixed during the shifting process. We are done because there are  $n - 3$  such triples in  $\mathcal{H}'$  that do not contain  $x'$ . However, it is possible that we reach Case (3) this time. Assume that we reach Case (3), say,  $\mathcal{H}' := S_{x'y'}(\mathcal{H}'')$  is HM at some 3-set  $A$  for some  $x' < y'$ . We claim that  $A = \{x', x, y\}$ . Indeed, since  $\{x, y\} = X_1$  is in at least  $n - 2 \geq 6$  edges of  $\mathcal{H}'$ , we know that  $\{x, y\} \subseteq A$  by Fact 2.5. Moreover, by Fact 2.3, we have  $x' \in A$  and thus  $A = \{x', x, y\}$ . Since  $\mathcal{H}''$  is not HM at  $\{x', x, y\}$ , we can pick an element  $z'$  such that  $\{x, y', z'\} \in \mathcal{H}''$  or  $\{y, y', z'\} \in \mathcal{H}''$ . We then set  $X_1 = \{x, y, y', z'\}$  and do the shift for all  $x'', y'' \in X \setminus X_1$ ,  $x'' < y''$ . The same arguments show that this time we will not reach Case (1) or (2). By similar reasons as before, the resulting family can only be HM at  $\{x'', x, y\}$ . This is also impossible because the family contains  $\{x, y', z'\}$  (or  $\{y, y', z'\}$ ).

Second assume that we reach Case (2). Let  $X_2 = \{x, y, z_1, z_2\}$ , where  $E_0 = \{z_1, z_2, z_3\}$  is defined as in the case  $k \geq 4$  and without loss of generality,  $z_1 \neq y$  and  $z_2 \neq y$ . Note that for any  $E \in \mathcal{H}_2 \setminus \{E_0\}$ ,  $E \cap \{x, y\} \neq \emptyset$ . So  $E \cap X_2 \neq \emptyset$  for all  $E \in \mathcal{H}_2$ , and moreover, by the maximality of  $|\mathcal{H}|$ , we may assume that  $\{x, y, z_1\}, \{x, y, z_2\} \in \mathcal{H}_2$ . We apply repeatedly  $S_{x'y'}$  to the family for  $x' < y'$ ,  $x', y' \in X \setminus X_2$ . In each step  $\mathcal{H}'$ , we know that  $\{x, y, z_1\}, \{x, y, z_2\} \in \mathcal{H}'$ , so we will not reach Case (1) or (2). Moreover, if  $\mathcal{H}' = S_{x'y'}(\mathcal{H}'')$  is HM at a 3-set  $A$ , then both  $\{x, y, z_1\}$  and  $\{x, y, z_2\}$  can miss at most one element of  $A$ . By Fact 2.3,  $x' \in A$  and thus  $A = \{x', x, y\}$ . Recall that  $E_0 = \{z_1, z_2, z_3\} \in \mathcal{H}_2$  and  $\{z_1, z_2\} \cap \{x', x, y\} = \emptyset$ . This implies that  $|E'_0 \cap \{x', x, y\}| \leq 1$ , where  $E'_0 \in \mathcal{H}'$  represents the set obtained from  $E_0$  after a series of shifts. This is a contradiction and thus we will not reach Case (3).

At last, assume that we reach Case (3). Let  $X_3 = \{x, y, x_1, x_2\}$ . Note that for any  $E \in \mathcal{H}_3$ ,  $|E \cap X_3| \geq 2$ . By the maximality of  $|\mathcal{H}|$ , we may assume that  $\{x, y, x_1\}, \{x, y, x_2\} \in \mathcal{H}_3$ . We apply repeatedly  $S_{x'y'}$  to the family for  $x' < y'$ ,  $x', y' \in X \setminus X_3$ . In each step  $\mathcal{H}'$ , we know that  $\{x, y, x_1\}, \{x, y, x_2\} \in \mathcal{H}'$ , so we will not reach Case (1) or (2). Moreover, assume that  $\mathcal{H}' = S_{x'y'}(\mathcal{H}'')$  is HM at a 3-set  $A$ . Since every set in  $\mathcal{H}_3$  contains  $x_1$  or  $x_2$ , by the maximality of  $|\mathcal{H}|$ , we

may assume that all sets containing  $x_1$  and  $x_2$  are in  $\mathcal{H}_3$  (so in  $\mathcal{H}'$ ). This implies that  $\{x_1, x_2\} \subseteq A$  by Fact 2.5. By Fact 2.3,  $x' \in A$  and thus  $A = \{x', x_1, x_2\}$ . However, since  $\{x, y, x_1\} \in \mathcal{H}'$  and  $|\{x, y, x_1\} \cap A| = 1$ , we get a contradiction. So we will not reach Case (3).

Let  $\mathcal{G}$  be the resulting family. Note that  $E \cap X_i \neq \emptyset$  for all  $E \in \mathcal{H}_i$  implies that  $E \cap X_i \neq \emptyset$  for all  $E \in \mathcal{G}$ , because the shifts do not affect elements in  $X_i$ .  $\square$

Eventually we obtain a family  $\mathcal{G}$  such that

- (i)  $E \cap X_i \neq \emptyset$  for all  $E \in \mathcal{G}$  and  $i = 1, 2, 3$ ,
- (ii)  $S_{x'y'}(\mathcal{G}) = \mathcal{G}$  for  $x' < y'$ ,  $x', y' \in X \setminus X_i$ .

Let  $X_0 = \emptyset$ . For  $k \geq 4$  and  $i = 0, 1, 2$ , let  $Y_i$  be the set of the first  $2k - |X_i|$  elements of  $X \setminus X_i$ . For  $k = 3$  and  $i = 0, 1, 2, 3$ , let  $Y_i$  be the set of the first  $7 - |X_i|$  elements of  $X \setminus X_i$ . By Proposition 2.4, in all cases, we have  $|Y_i| \geq 2k - 3$ . If we end up with Case (j) in Proposition 2.4 ( $j \in \{1, 2, 3\}$ ), let  $Y = X_j \cup Y_j$ , and thus  $|Y| = 2k$  or  $2k + 1$ .

**Lemma 2.6.** *For all  $E, E' \in \mathcal{G}$ ,  $E \cap E' \cap Y \neq \emptyset$  holds.*

*Proof.* First let  $i = 1, 2, 3$ . Suppose for a contradiction that  $E \cap E' \cap Y = \emptyset$  and  $E, E' \in \mathcal{G}$  such that  $|E \cap E'|$  is minimal. By (i) and  $|E \cap E' \cap (X \setminus Y)| \geq 1$ , we have

$$|(E \cup E') \cap Y_i| \leq |E \cap Y_i| + |E' \cap Y_i| \leq 2k - 4.$$

Since  $|Y_i| \geq 2k - 3$ , there is an element  $a \in Y_i \setminus (E \cup E')$ . Pick any  $b \in E \cap E' \cap (X \setminus Y)$  and note that  $a < b$ . By (ii), we know  $E'' := (E' \setminus \{b\}) \cup \{a\} \in \mathcal{G}$ . This is a contradiction because  $E \cap E'' \cap Y = \emptyset$  and  $|E \cap E''| < |E \cap E'|$ . The case  $Y = X_0 \cup Y_0$  is similar.  $\square$

For  $i \in [k]$ , let  $\mathcal{A}_i = \{E \cap Y : E \in \mathcal{G}, |E \cap Y| = i\}$ . By the definition of  $\mathcal{A}_i$  and Lemma 2.6, we have the following fact.

**Fact 2.7.** *The family  $(\bigcup_{1 \leq i \leq k} \mathcal{A}_i) \cup \mathcal{G}$  is intersecting.*

The following lemma is devoted to our final counting.

**Lemma 2.8.** *For  $k = 3$ , we have  $|\mathcal{A}_1| = 0$ ,  $|\mathcal{A}_2| \leq 2$  and  $|\mathcal{A}_3| \leq 12$ . For  $k \geq 4$ , we have*

$$|\mathcal{A}_i| \leq \binom{2k-1}{i-1} - \binom{k-1}{i-1} - \binom{k-2}{i-2} \text{ for } 1 \leq i \leq k-1$$

and

$$|\mathcal{A}_k| \leq \frac{1}{2} \binom{2k}{k} = \binom{2k-1}{k-1} - \binom{k-1}{k-1} - \binom{k-2}{k-2} + 2.$$

*Proof.* If  $|\mathcal{A}_1| > 0$ , then there is a set  $E \in \mathcal{G}$ , such that  $E \cap Y = \{x\}$  for some  $x \in Y$ . By Fact 2.2, there is a set  $E' \in \mathcal{G}$  such that  $x \notin E'$ . Thus  $E \cap E' \cap Y = \emptyset$ , contradicting Lemma 2.6. So  $|\mathcal{A}_1| = 0$  as desired.

First assume  $k = 3$ . Assume to the contrary that  $|\mathcal{A}_2| \geq 3$ . Since  $\mathcal{A}_2$  is 2-uniform and intersecting,  $\mathcal{A}_2$  is a star or a triangle. If  $\mathcal{A}_2$  is a star at  $x$  of size at least 3, then there is at most one triple that avoids  $x$  and meets each edge of the star, contradicting Fact 2.2. Otherwise  $\mathcal{A}_2$  is a triangle at  $\{x, y, z\}$ . In this case any member of  $\mathcal{A}_2 \cup \mathcal{A}_3$  (and thus any member of  $\mathcal{G}$ ) must contain at least two

elements of  $\{x, y, z\}$ , which means that  $\mathcal{G} \subseteq_S \mathcal{G}_2$ , a contradiction. Thus we get  $|\mathcal{A}_2| \leq 2$ . Note that  $\mathcal{A}_3$  is the induced subfamily of  $\mathcal{G}$  on  $Y$ , which is a 3-uniform family on 7 elements ( $|Y| = 2k+1 = 7$ ). Assume to the contrary that  $|\mathcal{A}_3| \geq 13$ . By Theorem 1.2, we know that  $\mathcal{A}_3$  is either EKR or HM. First assume that  $\mathcal{A}_3$  is EKR or HM at some  $x \in Y$ , i.e.,  $d_{\mathcal{A}_3}(x) \geq |\mathcal{A}_3| - 1$ . Since  $\mathcal{G}$  is neither EKR nor HM, there is an edge  $E \in \mathcal{G} \setminus \mathcal{A}_3$  such that  $x \notin E$ . Moreover, because  $\mathcal{A}_1 = \emptyset$ , we know that  $x \notin E \cap Y \in \mathcal{A}_2$ . Since  $\mathcal{A}_2 \cup \mathcal{A}_3$  is intersecting (Fact 2.7), every edge in  $\mathcal{A}_3$  must intersect  $E \cap Y$ . This implies that  $d_{\mathcal{A}_3}(x) \leq 9$ , and thus  $|\mathcal{A}_3| \leq d_{\mathcal{A}_3}(x) + 1 \leq 10$ , a contradiction. Second assume that  $\mathcal{A}_3$  is HM at some  $\{x, y, z\} \in Y$ , so  $|\mathcal{A}_3| = 13$ . Similarly, since  $\mathcal{G}$  is not HM, there is an edge  $E \in \mathcal{G} \setminus \mathcal{A}_3$  such that  $|E \cap \{x, y, z\}| \leq 1$ . Without loss of generality, assume that  $y, z \notin E$ . Let  $z' \in Y \setminus \{x, y, z\}$  and note that  $\{y, z, z'\} \in \mathcal{G}$ . Then  $E \cap \{y, z, z'\} = \emptyset$ , contradicting that  $\mathcal{G}$  is intersecting. So  $|\mathcal{A}_3| \leq 12$  holds.

Now assume  $k \geq 4$ . Fix  $2 \leq i \leq k-1$ . Observe that

$$\binom{2k-1-i}{i-1} - \binom{k-1}{i-1} = \binom{2k-2-i}{i-2} + \cdots + \binom{k-1}{i-2} \geq 2.$$

Indeed, since there are  $k-i$  binomial coefficients in the sum and each of them is at least 1, the inequality holds if  $i \leq k-2$ . Otherwise  $i = k-1$ , then  $\binom{2k-1-i}{i-1} - \binom{k-1}{i-1} = \binom{k-1}{2} \geq 2$  as  $k \geq 4$ . Assume that, to the contrary of the inequality in the lemma, we have

$$\begin{aligned} |\mathcal{A}_i| &> \binom{2k-1}{i-1} - \binom{k-1}{i-1} - \binom{k-2}{i-2} \\ &\geq \binom{2k-1}{i-1} - \binom{2k-1-i}{i-1} - \binom{2k-2-i}{i-2} + 2. \end{aligned}$$

Since  $\mathcal{A}_i$  is an intersecting  $i$ -uniform family on  $2k$  vertices, we may assume, by induction on  $i$ , that  $\mathcal{A}_i$  is EKR or HM at some  $x \in Y$ , or  $\mathcal{A}_i$  is HM at some  $\{x, y, z\} \subseteq Y$  for  $i = 3$ .

We first assume that  $\mathcal{A}_i$  is EKR or HM at some  $x$ . So  $\mathcal{A}_i$  contains at most one  $i$ -set  $A$  which does not contain  $x$ . Pick  $E, E' \in \mathcal{G}$  such that  $x \notin E$ ,  $x \notin E'$  and  $|E \cap Y|$  is minimal. Let  $|E \cap E' \cap Y| = t$ ,  $|(E \cap Y) \setminus E'| = t_1$  and  $|(E' \cap Y) \setminus E| = t_2$ . Clearly,  $1 \leq t \leq k-1$  and  $t + t_1 \leq t + t_2 \leq k$ . Since  $\mathcal{A}_i \cup \{E, E'\}$  is intersecting, we have

$$|\mathcal{A}_i| \leq \binom{2k-1}{i-1} - \binom{2k-1-t-t_1}{i-1} - \binom{2k-1-t-t_2}{i-1} + \binom{2k-1-t-t_1-t_2}{i-1} + c, \quad (2.1)$$

where  $c = 1$  if  $\mathcal{A}_i$  contains an  $i$ -set that does not contain  $x$  and  $c = 0$  otherwise. Note that

$$\begin{aligned} & - \binom{2k-1-t-t_1}{i-1} - \binom{2k-1-t-t_2}{i-1} + \binom{2k-1-t-t_1-t_2}{i-1} \\ &= - \binom{2k-1-t-t_1}{i-1} - \binom{2k-2-t-t_2}{i-2} - \cdots - \binom{2k-1-t-t_1-t_2}{i-2}. \end{aligned}$$

So in (2.1), we can substitute  $t_1$  and  $t_2$  by  $k-t$  (this will not decrease the bound), that is,

$$|\mathcal{A}_i| \leq \binom{2k-1}{i-1} - \binom{k-1}{i-1} - \binom{k-1}{i-1} + \binom{t-1}{i-1} + c.$$

Similarly, we can substitute  $t$  by  $k-1$ , that is,  $|\mathcal{A}_i| \leq \binom{2k-1}{i-1} - 2\binom{k-1}{i-1} + \binom{k-2}{i-1} + c$ . Moreover, the inequality is tight only if  $t + t_1 = k$ , but  $c = 1$  holds only if  $t + t_1 \leq i \leq k-1$ . Since we cannot



have both holding simultaneously, we have the desired bound

$$|\mathcal{A}_i| \leq \binom{2k-1}{i-1} - 2\binom{k-1}{i-1} + \binom{k-2}{i-1} = \binom{2k-1}{i-1} - \binom{k-1}{i-1} - \binom{k-2}{i-2}.$$

Next assume that  $i = 3$  and  $\mathcal{A}_i$  is HM at some  $\{x, y, z\} \subseteq Y$ . In this case it is easy to see that  $|\mathcal{A}_i| \leq 3(2k-3) + 1 = 6k-8$ . Note that when  $k \geq 4$ , we have  $6k-8 \leq \binom{2k-1}{i-1} - \binom{k-1}{i-1} - \binom{k-2}{i-2}$ .

At last, by Theorem 1.1, we have  $|\mathcal{A}_k| \leq \binom{2k-1}{k-1} = \binom{2k-1}{k-1} - \binom{k-1}{k-1} - \binom{k-2}{k-2} + 2$ .  $\square$

Now we proceed to the final estimation. Note that for a fixed  $A \in \mathcal{A}_i$ , there are at most  $\binom{n-|Y|}{k-i}$   $k$ -element sets  $E$  with  $E \cap Y = A$ . For  $k = 3$ , we get

$$|\mathcal{G}| \leq \sum_{i=1}^k |\mathcal{A}_i| \binom{n-2k-1}{k-i} \leq 2 \binom{n-2k-1}{k-2} + 12 = 2n-2, \quad (2.2)$$

as desired. For  $k \geq 4$ , we have

$$\begin{aligned} |\mathcal{G}| &\leq \sum_{i=1}^k |\mathcal{A}_i| \binom{n-2k}{k-i} \leq 2 + \sum_{i=1}^k \left( \binom{2k-1}{i-1} - \binom{k-1}{i-1} - \binom{k-2}{i-2} \right) \binom{n-2k}{k-i} \\ &= \binom{n-1}{k-1} - \binom{n-k-1}{k-1} - \binom{n-k-2}{k-2} + 2, \end{aligned} \quad (2.3)$$

proving the inequality part of the theorem.

### 3. THE UNIQUENESS IN THE THEOREM

**3.1. The stability of the shifts.** The following lemma shows the ‘stability’ of the shifts.

**Lemma 3.1.** *Let  $\mathcal{H}$  be a  $k$ -uniform intersecting family. If  $k \geq 3$  and  $S_{xy}(\mathcal{H}) = \mathcal{F}$  for some  $\mathcal{F} \in \{\mathcal{J}_2, \mathcal{G}_{k-1}, \mathcal{G}_2\}$ , then  $\mathcal{H}$  is isomorphic to  $\mathcal{F}$ .*

We first prove the following propositions. For two families  $\mathcal{A}_1$  and  $\mathcal{A}_2$  on the same set, we say  $(\mathcal{A}_1, \mathcal{A}_2)$  is *cross-intersecting* if for any  $A_1 \in \mathcal{A}_1$  and  $A_2 \in \mathcal{A}_2$ ,  $A_1 \cap A_2 \neq \emptyset$ . A family  $\mathcal{A}$  is called *non-separable* if for any partition  $\mathcal{A}_1 \cup \mathcal{A}_2$  of  $\mathcal{A}$  such that  $(\mathcal{A}_1, \mathcal{A}_2)$  is cross-intersecting, we have that  $\mathcal{A}_1 = \emptyset$  or  $\mathcal{A}_2 = \emptyset$ .

**Proposition 3.2.** *Fix  $s \geq 2$  and  $a$  and  $b \geq 1$  such that  $a+b \leq s$ . Let  $C$  be a set of size at least  $s+1$  and let  $z_1, z_2 \notin C$ . Let  $\mathcal{C}$  be the family on  $C \cup \{z_1, z_2\}$  such that*

$$\mathcal{C} = \{\{z_1\} \cup D : D \subseteq C, |D| = a\} \cup \{\{z_2\} \cup E : E \subseteq C, |E| = b\}.$$

*Then  $\mathcal{C}$  is non-separable.*

*Proof.* Consider a partition  $\mathcal{C} = \mathcal{C}' \cup \mathcal{C}''$ . Since  $|C| \geq s+1$ , for any two  $a$ -sets  $D_1, D_2$  in  $C$  such that  $|D_1 \cap D_2| = a-1$ , there is a  $b$ -set  $E \subseteq C \setminus (D_1 \cup D_2)$ . Thus both  $\{z_1\} \cup D_1$  and  $\{z_1\} \cup D_2$  are disjoint from  $\{z_2\} \cup E$  and thus they must belong to the same part. Observe that for any two  $a$ -sets  $D, D'$  in  $C$ , we can pick a sequence of  $a$ -sets  $D_1, \dots, D_t$  such that  $|D_i \cap D_{i+1}| = a-1$  for  $i \in [t-1]$  and  $|D \cap D_1| = |D' \cap D_t| = a-1$ . So all sets of form  $\{z_1\} \cup D$  are in the same part. Clearly, for any  $b$ -set  $E \subseteq C$ , there exists  $D \subseteq C$  such that  $\{z_2\} \cup E$  and  $\{z_1\} \cup D$  are disjoint and thus they must be in the same part. So all sets in  $\mathcal{C}$  are in the same part and we are done.  $\square$

**Proposition 3.3.** Fix  $r \geq 2$ . Let  $Z$  be a set of size  $m \geq 2r + 1$  and let  $A \subseteq Z$  such that  $|A| \in \{r - 1, r\}$ . Let  $\mathcal{B}$  be an  $r$ -uniform family on  $Z$  such that  $\mathcal{B} = \{B \subseteq Z : 0 < |B \cap A| < |A|\}$ . Then  $\mathcal{B}$  is non-separable.

*Proof.* First we assume  $r = 2$ . If  $|A| = 1$ , then  $\mathcal{B} = \emptyset$  and we are done. Otherwise  $|A| = 2$ . Note that in this case  $\mathcal{B}$  is isomorphic to the complete bipartite graph  $K_{2, m-2}$ , where  $m - 2 \geq 3$ . Then the proposition follows immediately from Proposition 3.2 by setting  $a = b = 1$ ,  $s = 2$  and  $C = Z \setminus A$ .

Now assume  $r \geq 3$ . Consider a partition  $\mathcal{B} = \mathcal{B}' \cup \mathcal{B}''$ . Let  $\mathcal{B}_i = \{E \in \mathcal{B} : |E \cap A| = i\}$  for all  $1 \leq i \leq r - 1$ .

**Claim.**  $\mathcal{B}_1$  is non-separable.

Note that the claim implies the proposition. Indeed, without loss of generality, assume that  $\mathcal{B}_1 \subseteq \mathcal{B}'$ . Note that  $\mathcal{B}_{r-1} = \emptyset$  when  $|A| = r - 1$ . For any set  $E$  in  $\mathcal{B}_{r-1}$  or  $\mathcal{B}_{r-2}$ , we can find a set  $E'$  in  $\mathcal{B}_1$  that is disjoint from  $E$ . So  $E$  and thus all sets in  $\mathcal{B}_{r-1}$  and  $\mathcal{B}_{r-2}$  must be in  $\mathcal{B}'$ . Similarly, using the sets in  $\mathcal{B}_{r-2}$ , we conclude that all sets in  $\mathcal{B}_2$  must be in  $\mathcal{B}'$ . So we get  $\mathcal{B} = \mathcal{B}'$  after iteratively applying the same arguments.

So it remains to prove the claim.

*Proof of the claim.* Let  $C = Z \setminus A$ . We first prove the case when  $|A| = r$ , say,  $A = \{x_1, \dots, x_r\}$ . Note that  $|C| = m - r \geq r + 1$ . Let  $a = 1$ ,  $b = r - 1$  and  $s = r$ . We replace  $\{x_1, \dots, x_{r-1}\}$  by  $z_1$  and set  $z_2 = x_r$ . Applying Proposition 3.2 shows that all sets in  $\mathcal{B}_{r-1}$  that contain  $\{x_1, \dots, x_{r-1}\}$  and all sets in  $\mathcal{B}_1$  that contain  $x_r$  are in the same part, say,  $\mathcal{B}'$ . Note that this implies that all sets in  $\mathcal{B}_{r-2}$  that do not contain  $x_r$  are in  $\mathcal{B}'$ . Now fix any set  $F \in \mathcal{B}_1$ , let  $F \cap A = \{x_i\}$  for some  $i \in [r]$ . Since  $|A| = r$  and  $|C| \geq r + 1$ , we can always pick a set in  $\mathcal{B}_{r-2}$  that do not contain  $x_r$  and is disjoint from  $F$ . So  $F$  and thus all sets in  $\mathcal{B}_1$  are in  $\mathcal{B}'$ .

Next assume  $|A| = r - 1$ , say,  $A = \{x_1, \dots, x_{r-1}\}$ . Note that  $|C| = m - (r - 1) \geq r + 2$ . Let  $a = 2$ ,  $b = r - 1$  and  $s = r + 1$ . We replace  $\{x_1, \dots, x_{r-2}\}$  by  $z_1$  and set  $z_2 = x_{r-1}$ . Applying Proposition 3.2 shows that all sets in  $\mathcal{B}_{r-2}$  that contain  $\{x_1, \dots, x_{r-2}\}$  and all sets in  $\mathcal{B}_1$  that contain  $x_{r-1}$  are in the same part, say,  $\mathcal{B}'$ . Note that if  $r = 3$ , then we are done because  $\mathcal{B} = \mathcal{B}_1$  and any set in  $\mathcal{B}_1$  contains exactly one of  $x_1$  and  $x_2$ . Otherwise,  $r \geq 4$ . Note that all sets in  $\mathcal{B}_{r-3}$  that do not contain  $x_{r-1}$  are in  $\mathcal{B}'$ . Now fix any set  $F \in \mathcal{B}_1$ , let  $F \cap A = \{x_i\}$  for some  $i \in [r - 1]$ . Since  $|A| = r - 1$  and  $|C| \geq r + 2$ , we can always pick a set in  $\mathcal{B}_{r-3}$  that do not contain  $x_{r-1}$  and is disjoint from  $F$ . So  $F$  and thus all sets in  $\mathcal{B}_1$  are in  $\mathcal{B}'$ . Thus the proof of the claim is complete.  $\square$

Now we show Lemma 3.1. For a family  $\mathcal{C}$  and an element  $z$ , let  $\mathcal{C}(z) = \{E \cup \{z\} : E \in \mathcal{C}\}$ . Our scheme is as follows. For the shift  $S_{xy} : \mathcal{H} \rightarrow \mathcal{F}$ , let  $\mathcal{B}_x$  be the subfamily of  $\mathcal{F}$  such that

$$\mathcal{B}_x = \{E \in \mathcal{F} : x \in E, y \notin E, (E \setminus \{x\}) \cup \{y\} \notin \mathcal{F}\}$$

and let  $\mathcal{B} = \{E \setminus \{x\} : E \in \mathcal{B}_x\}$ , which is a  $(k - 1)$ -uniform family. Clearly, only the sets in  $\mathcal{B}_x$  might be obtained from the shift  $S_{xy}$ , i.e.,  $\mathcal{F} \setminus \mathcal{B}_x \subseteq \mathcal{H}$ . So the shift  $S_{xy} : \mathcal{H} \rightarrow \mathcal{F}$  can be interpreted

as a partition  $\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2$  such that

$$\mathcal{H} = (\mathcal{F} \setminus \mathcal{B}_x) \cup \mathcal{B}_1(y) \cup \mathcal{B}_2(x) \text{ and } \mathcal{F} = (\mathcal{F} \setminus \mathcal{B}_x) \cup \mathcal{B}_1(x) \cup \mathcal{B}_2(x),$$

i.e., the sets in  $\mathcal{B}_1(y)$  are shifted to  $\mathcal{B}_1(x)$  by  $S_{xy}$ . Here a natural requirement is that  $\mathcal{B}_1(y) \cup \mathcal{B}_2(x)$  should be intersecting, i.e.,  $(\mathcal{B}_1, \mathcal{B}_2)$  should be cross-intersecting. We will show that  $\mathcal{B}$  is non-separable, i.e., any cross-intersecting partition  $\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2$  satisfies that  $\mathcal{B} = \mathcal{B}_1$  or  $\mathcal{B} = \mathcal{B}_2$ . Observe that in all of our cases,  $\mathcal{B} = \mathcal{B}_2$  means that  $\mathcal{H} = \mathcal{F}$ , and  $\mathcal{B} = \mathcal{B}_1$  means that  $\mathcal{H}$  is isomorphic to  $\mathcal{F}$  with  $y$  playing the role of  $x$ . This will conclude the proof.

We mention that, in most cases, we will apply Proposition 3.3 as follows. Let  $Z = X \setminus \{x, y\}$  and  $r = k - 1$ . Note that  $|Z| \geq 2k + 1 - 2 = 2r + 1$ . Our goal is to define  $A$  appropriately so that we can apply Proposition 3.3 and then conclude that  $\mathcal{B}$  is non-separable. We call the shift  $S_{xy} : \mathcal{H} \rightarrow \mathcal{F}$  *trivial* if  $\mathcal{B}_x = \emptyset$ .

*Proof of Lemma 3.1.* The proof consists of three cases on  $\mathcal{F}$ .

**Case 1.**  $\mathcal{F} = \mathcal{G}_{k-1}$ . We may assume  $k \geq 4$ , since when  $k = 3$ ,  $\mathcal{G}_{k-1} = \mathcal{G}_2$ , which will be solved in Case 2. We use the notation for  $\mathcal{G}_{k-1}$  in Definition 1.5: for a  $(k - 1)$ -set  $E \subset X$  and  $x_0 \in X \setminus E$ , let  $\mathcal{G}_{k-1}$  be the  $k$ -uniform family such that

$$\mathcal{G}_{k-1} = \{G : E \subseteq G\} \cup \{G : x_0 \in G, G \cap E \neq \emptyset\}.$$

The family  $\mathcal{G}_{k-1}$  partitions  $X$  into three types of elements:

- Type 1:  $T_1 = \{x_0\}$ ,
- Type 2:  $T_2 = E$  and
- Type 3: the set  $T_3$  of the remaining elements ( $|T_3| = n - k \geq k + 1$ ).

Observe that shifts between two elements of the same type are trivial and for any  $i < j$ , any shift from an element of  $T_i$  to an element of  $T_j$  is trivial. So we have the following two cases.

We first assume  $x = x_0$ . Let  $Z = X \setminus \{x, y\}$ ,  $r = k - 1 \geq 3$  and  $A = E \setminus \{y\}$ . Note that  $|A| \in \{r - 1, r\}$  and we have  $\mathcal{B} = \{B \subseteq Z : 0 < |B \cap A| < |A|\}$ . So we can apply Proposition 3.3 and the proof of this case is finished.

Next we assume that  $x \in T_2$  and  $y \in T_3$ . Let  $C = T_3 \setminus \{y\}$ , we have

$$\mathcal{B} = \{(E \setminus \{x\}) \cup \{z\} : z \in C\} \cup \{\{x_0\} \cup F : F \subseteq C, |F| = k - 2\}.$$

Let  $a = 1$ ,  $b = k - 2$  and  $s = k - 1$ . We replace  $E \setminus \{x\}$  by  $z_1$  and set  $z_2 = x_0$ . Applying Proposition 3.2 concludes the proof.

**Case 2.**  $\mathcal{F} = \mathcal{G}_2$ . Observe that  $\mathcal{G}_2$  contains only two types of elements: the elements in the 3-set, and the other elements. Observe that the shifts between two elements of the same type are trivial. Also, if  $y$  is in the 3-set, then the shift is trivial. So assume that the 3-set is  $\{x, x_1, x_2\}$  and  $y \notin \{x, x_1, x_2\}$ . Let  $C = X \setminus \{x, x_1, x_2, y\}$ ,  $a = b = k - 2$  and  $s = 2k - 2$ . So we have

$$\mathcal{B} = \{\{x_1\} \cup D : D \subseteq C, |D| = a\} \cup \{\{x_2\} \cup E : E \subseteq C, |E| = b\}$$

and we conclude this case by Proposition 3.2.

**Case 3.**  $\mathcal{F} = \mathcal{J}_2$ . We use the notation for  $\mathcal{J}_2$  in Definition 1.5: for a  $(k-1)$ -set  $E \subseteq X$  and a 3-set  $J = \{x_0, x_1, x_2\} \subseteq X \setminus E$ , let  $\mathcal{J}_2$  be a  $k$ -uniform family such that

$$\mathcal{J}_2 = \{G : E \subseteq G, G \cap J \neq \emptyset\} \cup \{G : J \subseteq G\} \cup \{G : x_0 \in G, G \cap E \neq \emptyset\}.$$

The family  $\mathcal{J}_2$  partitions  $X$  into four types of elements:

- Type 1:  $T_1 = \{x_0\}$ ,
- Type 2:  $T_2 = E$ ,
- Type 3:  $T_3 = \{x_1, x_2\}$  and
- Type 4: the set  $T_4$  of the remaining elements ( $|T_4| = n - k - 2 \geq k - 1$ ).

Observe that shifts between two elements of the same type are trivial and for any  $i < j$ , any shift from an element of  $T_i$  to another element of  $T_j$  is trivial.

We first assume  $x = x_0$ . If  $y \in T_3$ , let  $A = E$  and note that  $\mathcal{B} = \{B \subseteq Z : 0 < |B \cap A| < |A|\}$ . So we can apply Proposition 3.3 and conclude the proof of this case. Otherwise,  $y \in T_2 \cup T_4$ . This case is more involved. Let  $A = E \setminus \{y\}$  and thus  $|A| \in \{r-1, r\}$ . Consider a cross-intersecting partition  $\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2$ . Observe that  $\{B \subseteq Z : 0 < |B \cap A| < |A|\} \subseteq \mathcal{B}$ . We partition  $\mathcal{B}$  into three subfamilies  $\mathcal{B}^*, \mathcal{B}^0, \mathcal{B}^{**}$ , where

$$\begin{aligned} \mathcal{B}^* &= \{B \subseteq Z : 0 < |B \cap A| < |A|\}, \\ \mathcal{B}^0 &= \{B \in \mathcal{B} : B \cap A = \emptyset\} = \{\{x_1, x_2\} \cup F : F \subseteq T_4 \setminus \{y\}, |F| = k-3\} \text{ and} \\ \mathcal{B}^{**} &= \{B \in \mathcal{B} : A \subseteq B\} = \begin{cases} \{A \cup \{z\} : z \in T_4\} & \text{if } y \in T_2, \\ \{A\} & \text{if } y \in T_4. \end{cases} \end{aligned}$$

To get a cross-intersecting partition of  $\mathcal{B}$ , we need to distribute the sets in  $\mathcal{B}^* \cup \mathcal{B}^0 \cup \mathcal{B}^{**}$ . Note that  $\mathcal{B}^* = \emptyset$  if and only if  $|A| = 1$ , which, in turn, is equivalent to  $k = 3$  and  $y \in T_2$ . In this case  $\mathcal{B}^0$  contains only  $\{x_1, x_2\}$ , which is disjoint from all other sets in  $\mathcal{B}^{**}$  and thus  $\mathcal{B}$  is non-separable.

Now suppose  $\mathcal{B}^* \neq \emptyset$ . We apply Proposition 3.3 and get that  $\mathcal{B}^*$  is non-separable, i.e.,  $\mathcal{B}^* \subseteq \mathcal{B}_1$  or  $\mathcal{B}^* \subseteq \mathcal{B}_2$ . We first consider any set  $P \in \mathcal{B}^0$ . Fix any  $a \in A$ . Since  $|Z \setminus \{a\}| \geq 2r$ , there exists an  $r$ -set  $P' \subseteq Z \setminus (\{a\} \cup P)$  such that  $0 < |P' \cap A| < |A|$  and  $P \cap P' = \emptyset$ . Thus,  $P \in \mathcal{B}^0$  and  $P' \in \mathcal{B}^*$  must be in the same part. So all sets of  $\mathcal{B}^* \cup \mathcal{B}^0$  belong to the same part. Next, consider any set  $B \in \mathcal{B}^{**}$  and note that  $|B \cap T_4| \leq 1$ . Clearly, since  $|T_4| \geq k-1$ , there exists a  $(k-3)$ -set  $F \subseteq T_4 \setminus \{y\}$  such that  $B \cap F = \emptyset$ . So  $P = F \cup \{x_1, x_2\} \in \mathcal{B}^0$  and  $B \cap P = \emptyset$ , which implies that  $P$  and  $B$  belong to the same part. Thus, we conclude that  $\mathcal{B} = \mathcal{B}_1$  or  $\mathcal{B} = \mathcal{B}_2$  and we are done.

Next we assume  $x \in T_2$ . Let  $E_i = (E \cup \{x_i\}) \setminus \{x\}$  for  $i = 1, 2$ . Observe that if  $y \in T_4$ , then

$$\mathcal{B} = \{E_1, E_2\} \cup \{G : x_0 \in G, G \cap E = \emptyset, |G \cap \{x_1, x_2\}| \leq 1, y \notin G\}.$$

Otherwise  $y \in T_3$ , without loss of generality,  $y = x_1$ , then

$$\mathcal{B} = \{E_2\} \cup \{G : x_0 \in G, G \cap E = \emptyset, G \cap \{x_1, x_2\} = \emptyset\}.$$

In the former case, since  $|T_4 \setminus \{y\}| \geq k-2$ , there is a set  $B \in \mathcal{B} \setminus \{E_1, E_2\}$  such that  $B \cap \{x_1, x_2\} = \emptyset$ . Note that  $B \cap E_1 = B \cap E_2 = \emptyset$ , so  $E_1, E_2$  must belong to the same part. Moreover, for any set  $B' \in \mathcal{B} \setminus \{E_1, E_2\}$ , because  $|B' \cap \{x_1, x_2\}| \leq 1$ , we have  $B' \cap E_1 = \emptyset$  or  $B' \cap E_2 = \emptyset$ . Thus  $\mathcal{B}$  is

non-separable and we are done. In the latter case, observe that for any  $B \in \mathcal{B} \setminus \{E_2\}$ ,  $E_2 \cap B = \emptyset$ . So all sets in  $\mathcal{B}$  must belong to the part in which  $E_2$  is and we are done.

At last, we assume that  $x \in T_3$  and  $y \in T_4$ . Without loss of generality, let  $x = x_1$ . In this case, we have that

$$\mathcal{B} = \{E\} \cup \{G : \{x_0, x_2\} \subseteq G, G \cap E = \emptyset, y \notin G\}.$$

Clearly, for any  $B \in \mathcal{B} \setminus \{E\}$ ,  $E \cap B = \emptyset$ . So all sets in  $\mathcal{B}$  must belong to the part in which  $E$  is and we are done.  $\square$

**3.2. The case  $n \geq 8$  for  $k = 3$  and  $n \geq 2k + 1$  for  $k \geq 4$ .** We assume that the equality in (2.2) or (2.3) holds. We first show that  $\mathcal{G}$ , the family obtained by the shifts, is isomorphic to one of the extremal examples. Indeed, to have equality we must have equality in Lemma 2.8. Note that  $|\mathcal{A}_2| = k - 1$  implies that  $\mathcal{A}_2$  is a star of size  $k - 1$  or a triangle (for  $k = 4$  only). If  $k = 4$  and  $\mathcal{A}_2$  is a triangle at  $\{x, y, z\}$ , then any member of  $\bigcup_{1 \leq i \leq k} \mathcal{A}_i$  (and thus any member of  $\mathcal{G}$ ) must contain at least two elements of  $\{x, y, z\}$ , which means that  $\mathcal{G} \subseteq_S \mathcal{G}_2$ . Otherwise, suppose  $\mathcal{A}_2$  is a star at  $x$  of size  $k - 1$ , say  $\{xx_1, \dots, xx_{k-1}\}$ . Note that all sets in  $\mathcal{G}$  not containing  $x$  must contain  $\{x_1, \dots, x_{k-1}\}$ . Moreover, there are at least two such sets by Fact 2.2. So according to the number of such sets in  $\mathcal{G}$ , we have  $\mathcal{G} \subseteq_S \mathcal{G}_{k-1}$  (only for  $k \geq 4$ ) or  $\mathcal{G} \subseteq_S \mathcal{J}_i$  for some  $2 \leq i \leq k - 1$ . Straightforward calculations show that the extremal value of  $|\mathcal{G}|$  is achieved by  $|\mathcal{G}_3|$  or  $|\mathcal{J}_2|$  when  $k = 4$ , and by  $|\mathcal{J}_2|$  only when  $k \neq 4$ . Then we are done by Lemma 3.1.

**3.3. The case  $n = 7$  and  $k = 3$ .** In this case we have to go through the shifting argument again. We assume that  $\mathcal{H}$  is of the maximal size subject to the assumptions, i.e.,  $|\mathcal{H}| = 12$ . Recall that if we apply  $S_{xy}$  repeatedly to  $\mathcal{H}$ , then we obtain a family in one of the following four cases,

- (0) a family  $\mathcal{G}$  which is stable, i.e.,  $S_{xy}(\mathcal{G}) = \mathcal{G}$  holds for all  $x < y$ ,
- (1) a family  $\mathcal{H}_1$  such that  $S_{xy}(\mathcal{H}_1)$  is EKR at  $x$ ,
- (2) a family  $\mathcal{H}_2$  such that  $\mathcal{G}' = S_{xy}(\mathcal{H}_2)$  is HM at  $x$ , or
- (3) a family  $\mathcal{H}_3$  such that  $\mathcal{G}'' = S_{xy}(\mathcal{H}_3)$  is HM at  $\{x, x_1, x_2\}$  for some  $x_1, x_2 \in X \setminus \{x, y\}$ .

We will use the following fact, the proof of which is straightforward and is thus omitted.

**Fact 3.4.** *Let  $\mathcal{D}_0$  be the graph  $K_{2,3}$  and let  $\mathcal{D}_1$  be the graph obtained from deleting any edge of  $\mathcal{D}_0$ . Let  $\mathcal{D}$  be the set of graphs which are obtained from deleting any two edges of  $K_5$ . Then  $\mathcal{D}_0, \mathcal{D}_1$  and all families in  $\mathcal{D}$  are non-separable.*

First assume that we reach Case (2) and note that  $|\mathcal{G}'| = 12 = |\mathcal{F}_1| - 1$ . We use the following notation. Let  $F$  be a 3-set of  $X$  and  $x \in X \setminus F$ . Let

$$\mathcal{F}_1 := \{F\} \cup \{G \subseteq X : x \in G, F \cap G \neq \emptyset\}.$$

Note that if  $y \in F$ , then  $\mathcal{B}$  is isomorphic to  $\mathcal{D}_0$  or  $\mathcal{D}_1$  and if  $y \notin F$ , then  $\mathcal{B}$  is isomorphic to some  $\mathcal{D}_2 \in \mathcal{D}$ . By Fact 3.4, in either case, we know that  $\mathcal{H}_2 \subseteq \mathcal{F}_1$ , a contradiction.

Second assume that we reach Case (3) and note that  $|\mathcal{G}''| = 12 = |\mathcal{G}_2| - 1$ . Since  $\mathcal{G}''$  misses only one set of  $\mathcal{G}_2$ , we know  $\{x, x_1, x_2\} \in \mathcal{G}''$  – otherwise  $\{y, x_1, x_2\}$  would have been shifted to  $\{x, x_1, x_2\}$ , a contradiction. Observe that if  $\{y, x_1, x_2\} \in \mathcal{G}''$ , then  $\mathcal{B}$  is isomorphic to  $\mathcal{D}_0$  or  $\mathcal{D}_1$ .

By Fact 3.4, in either case, we know that  $\mathcal{H}_3 \subseteq \mathcal{G}_2$ , a contradiction. Otherwise,  $\{x_1, x_2\} \in \mathcal{B}$  and  $\mathcal{B} \setminus \{x_1, x_2\}$  is isomorphic to  $\mathcal{D}_0$ . So in this case  $\mathcal{B}$  is not non-separable. However, since  $\mathcal{D}_0$  is non-separable, the only non-trivial partition of  $\mathcal{B}$  is  $\{\{x_1, x_2\}\}$  and  $\mathcal{B} \setminus \{x_1, x_2\}$ . In all cases (with trivial partitions or the non-trivial partition of  $\mathcal{B}$ ), it is easy to see that  $\mathcal{H}_3 \subseteq \mathcal{G}_2$ , a contradiction.

Finally we assume that we reach Case (1). Let  $X_1 = \{x, y\}$ . We apply the shifts  $S_{x'y'}$  for  $x', y' \in X \setminus X_1$  and  $x' < y'$ . Similar arguments in Proposition 2.4 show that we will not reach Case (1) again and by the previous two cases, we will not reach any of Case (2) or (3). So we must get a family  $\mathcal{G}$  such that  $S_{x'y'}(\mathcal{G}) = \mathcal{G}$  for all  $x', y' \in X \setminus X_1$ ,  $x' < y'$ .

Let  $X_0 = \emptyset$  and for  $i = 0, 1$ , let  $Y$  be the union of  $X_i$  and the first  $6 - |X_i|$  elements in  $X \setminus X_i$ . Note that we have  $|Y \setminus X_i| \geq 3 = 2k - 3$  and thus Lemma 2.6 holds. Then the proof of Lemma 2.8 gives that  $|\mathcal{A}_1| = 0$ ,  $|\mathcal{A}_2| \leq 2$  and  $|\mathcal{A}_3| \leq \binom{5}{2} = 10$ . So we get  $|\mathcal{G}| = |\mathcal{A}_2| + |\mathcal{A}_3| \leq 12$ . To have equality we must have  $|\mathcal{A}_2| = 2$  and the same argument in Section 3.2 implies that  $\mathcal{G} = \mathcal{J}_2$ . By Lemma 3.1,  $\mathcal{H} = \mathcal{J}_2$ .

#### 4. CONCLUDING REMARKS

We have restricted ourselves to intersecting families in this note, and did not consider  $t$ -intersecting families (that is, families in which the intersection of any two of its members has at least  $t$  elements) for  $t > 1$ . It would be natural to investigate this more general setting. Naturally, the starting point would be the generalization of the Hilton–Milner theorem to  $t$ -intersecting families [1] (see also [2]).

One may also consider going beyond Theorem 1.6: that is, *what is the maximum size of an intersecting family  $\mathcal{H}$  that is neither EKR, nor HM, nor is contained in  $\mathcal{J}_2$  (and in  $\mathcal{G}_2$  and  $\mathcal{G}_3$  if  $k = 4$ )?* Very recently, Kostochka and Mubayi [17] established that the answer is  $|\mathcal{J}_3|$  for all large enough  $n$ <sup>2</sup>. Naturally, it would be good to know the complete answer, that is, for all  $n$  and  $k$  with  $n > 2k$ .

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